**LINEAR ALGEBRA**

**BASICS OF MATRIX**

**Matrix Multiplication:** Let ,

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| 1. exist 2. exist 3. (Not Cumulative) 4. (Associative) | 1. involves No. of multiplications = And Number of Additions = Where, . |

**Trace of Matrix:** Let Matrix,

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| **Properties:**  The Principle Diagonal = () |  |

**Diagonal Matrix:** If is diagonal Matrix, .

Diagonal Matrix Denoted By **Note:**

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| **Upper Triangular Matrix:** In , | **Lower Triangular Matrix:** In , |

**Inverse of Matrix:**

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| 1. If , A is called **Singular Matrix**. And If , A is called **Non-Singular Matrix.** 2. . | 1. If , then is called the inverse of . |
| **Properties:** |  |

**Determinant of Matrix ():** Sum of Product of any Row/ Column elements and corresponding cofactors.

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| **Properties:**   1. If matrix has Zero Row/ Column, . 2. If two Row/ Column of matrix are equal/ proportional, . 3. If two Row / Column of the matrix interchanged, . 4. = Product of diagonal elements of Upper/ Lower Triangular Matrix only. | 1. If all elements of Row / Column are scalar multiple of K then, 2. If all elements of Matrix are multiplied by scalar multiple (K) then, 3. If every element of a Row/ Column is multiplied by a scalar and added to another Row/ Column, then the determinant remains same. |

**RANK OF MATRIX**

**RANK OF MATRIX:** Let be any zero matrix of order ,

**MINOR:** Determinant of Sub-matrix (Square matrix).

**SUB-MATRIX:** matrix Obtained by deleting rows and columns.

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| Properties:   1. If else . | 1. If else If . 2. If     . |

**ROW ECHELON FORM OF MATRIX:** Let Matrix is said to be in Row Echelon Form if,

1. Zero Rows (if any) should be below the non-zero rows.
2. Zero before first non-zero number in row should be less than zeros before first non-zero number in next row.

**Note-I:** To get Row Echelon Form perform Row Operations only.

**Note-II:** Rank of the Matrix is not affected by elementary row operations.

**SYSTEM OF LINER EQUATIONS**

**Matrix Form:**

**Solution of System of Equations:** The values of vector satisfies .

**Consistent System:** The System has at least one solution.

**In-Consistent System:** The System has no solution.

**Homogeneous System of Equation:** .

**Non-Homogeneous System of Equation:** .

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| **METHOD TO SOLVE SYSTEM OF EQUATIONS (m = n = 3)** | |
| Matrix Inversion Method | Cramer’s Method |
|  |  |

**SOLUTIONS:**

1. **Unique Solution:**
2. **Infinitely many solutions:**
3. **No Solution:**

**DISADVANTAGES:**

1. Applicable only for m = n.
2. Inverse Method Fails When
3. In Cramer’s Method We need to calculate *(n+1)* determinants of order *n*. hence, No suitable for *n>4.*

**GAUSSIAN ELIMINATION METHOD:** Let be the given system of linear equation.

**AUGMENTED MATRIX:** = A and B together.

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| Consistent System of Equation: | | Inconsistent System of Equation: |
|  |  | **No Solutions** |
| **Unique Solution** | **Infinitely many solutions** |  |

**HOMOGENEOUS SYSTEM OF EQUATIONS:**

|  |  |
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| **Trivial/ Zero Solution:** | **Non-Trivial/ Non-Zero Solution:** |

**Note: Every Homogeneous system is always consistent. But Non-Trivial solutions may or may not exists, if exists infinitely many solutions exists.**

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| And | And | 1. Always possesses infinitely many Non-Trivial solutions.   If , Only Trivial Solution  If , Infinitely many Non-Trivial solutions. |
| Only Trivial Solution | Infinitely many Non-Trivial solutions |

**NULL SPACE:** Set of all Solutions of .

**NULLITY:** Dimensions of null Space.

Where r= rank of matrix, and n = number of variables.

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| **Linearly Dependent:** | **Linearly Independent:** is not proportional to |

**EIGEN VALUES AND EIGEN VECTORS**

**CHARACTERISTIC EQUATIONS:** let Matrix, then is called characteristic equations of A.

**EIGEN VALUES:** Roots of characteristic equations is called Eigen Values.

**CAYLEY-HAMILTON THEOREM:** Every Square Matrix of order (n>1) satisfies it’s own characteristic equation.

**ADVANTAGES OF THEOREM:** Easily find . And can be express in lower power of A and I.

**ALGEBRAIC MULTIPLICITY OF EIGEN VALUE (λ):** No. of times eigen value occurred.

**EIGEN VECTOR:** A Non-Zero vector (X) is said to be eigen vector corresponds to the eigen value (λ) of the matrix (A) if .

**NOTE:**

1. Corresponding to one eigen value infinitely many eigen vector exists.
2. **GEOMETRIC MULTIPLICITY** = No. of linearly independent eigen vectors = .
3. **.**

**PROPERTY:** If is eigen value of non-singular matrix (), **1)** 1/ λ is eigen value of is eigen value of .

**DIAGONALIZATION:** Given Matrix () is said to be diagonalizable if there exists a non-singular matrix such that, .

Here Matrix with columns are eigen values of and Diagonal matrix with eigen values of as elements.

**PROPERTY:**

1. .
2. is diagonalizable . Eg. had linearly in-depend solutions.

**PROPERTY OF EIGEN VALUES AND EIGEN VECTORS:**

1. Sum of the eigen values = Trace of the vector
2. Product of the eigen values = determinant of the vector.
3. Zero is one of the eigen value of the matrix () and .
4. Eigen values of are same.
5. The eigen values of Upper/ Lower Triangular/Diagonal Matrix are just diagonal elements of Matrix only.
6. If is eigen value of the matrix, is also eigen value. **[COMPLEX RULE]**
7. If is eigen value of the matrix, is also eigen value. **[CONJUGATE RULE]**
8. Eigen vectors corresponding to distinct eigen values are linearly independent.
9. For different values of eigen value and Corresponding Eigen Vectors following rules applicable.
   1. 1/ λ is eigen value of .
   2. is eigen value of .
   3. is eigen value of.
   4. , then is eigen value.

**MODEL-I:** For given matrix, find eigen value.

**MODEL-II:** For given matrix and eigen value/vector, find eigen vector/vector.

**MODEL-III:** For givenEigen value And Eigen vectors, find matrix.

**MODEL-IV:** Find Values using Cayley-Hamilton Theorem.

**MODEL-V:** Problem Related to properties.

**SPECIAL MATRICES**

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| **Purely Real Number:** | **Purely Imaginary Number:** |

Conjugate of a complex number is Mirror of the point about X axis.

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| **Symmetric Matrix:** | **Hermitian Matrix:** |
| **Skew-Symmetric Matrix:** | **Skew-Hermitian Matrix:** |
| **Orthogonal Matrix:** | **Unitary Matrix:** |

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| **OBSERVATIONS:** | 1. Diagonal Element of Hermitian Matrix are real number. 2. Diagonal Element of Skew-Symmetric Matrix are Zero. 3. Diagonal Element of Skew- Hermitian Matrix are Zero. |

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| **RESULTS:** | 1. Let Matrix, can be expressed as sum of symmetric and Skew symmetric matrices. 2. Let Matrix, can be expressed as sum of Hermitian and Skew Hermitian matrices. 3. If A is an orthogonal Matrix, the . 4. If A is an orthogonal Matrix, the . |

**CONJUGATE MATRIX ():**

**CONJUGATE TRANSPOSE MATRIX ():**

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| **PROPERTY:**  1.  2. |  |

**POSITIVE INTEGER POWER OF :**

**IDEMPOTENT MATRIX:**

**NILPOTENT MATRIX:** where Index of Nilpotent matrix (least positive integer).

**INVOLUTORY MATRIX:** Where Square Matrix

**PERIODIC MATRIX:** Where = Periodicity (least positive integer)

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| **MATRIX** | **EIGEN VALUE** |
| **Hermitian Matrix** | Always Real |
| **Skew-Hermitian Matrix** | Either Zero or Pure Imaginary |
| **Orthogonal/ Unitary Matrix** | And can be Real or Complex conjugate |
| **Idempotent Matrix** | 0 or 1 |
| **Nilpotent Matrix** | 0 |
| **Involutory Matrix** |  |

**RESULTS:** Let be an orthogonal matrix. If is eigen value, is also eigen value.

**Orthogonal Vectors ():** Two vectors are said to be orthogonal if .

**Set of** **Orthogonal Vectors:** , where . Every pair is orthogonal.

**Norm of Vector:**

**Normalized Vector:**

**Set of** **Orthonormal Vectors:** , where and .

Or If .

**Result:** If A is an orthogonal matrix, then it’s rows/ columns are orthonormal.

**LU DECOMPOSITION**

**OBJECTIVE:** 1) Solving system of linear equations, 2) Finding , 3)

In , Each matrix contains unknowns/ Variables. hence total (L+U) has unknowns.

By comparing we get equations.

**DOOLITTLE’S METHOD:**

**CROUT’S METHOD:**

**CHOLESKY METHOD:** If a is symmetric matrix then,

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| **Note:** | 1. LU Decomposition fails if any of the diagonal elements of L or U is zero. 2. LU Decomposition Exists if the matrix is positive definite. |

**POSITIVE DEFINITE:** is said to be positive definite if all loading minors of a are positive.

Gauss Elimination method operation involves

LU Decomposition method operation involves

**Method LU:** First find forward substitute (UX=Z and LZ=B) (U) after that backward substitute(L).

**RESULT:** If a is non singular matrix, then can be obtained by row addition operation.

Where, L = Lower triangular matrix with diagonal element 1 And U = Upper triangular matrix.

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| **MINIMAL POLYNOMIAL:** | 1. If , then we say that the polynomial is annihilates the matrix A. 2. **Monic Polynomial:** The Coefficient of highest power of x is unity.   **Minimal polynomial:** 1) Lowest Degree monic polynomial 2) annihilates the matrix A. |